INTEGRABILITY VIA REVERSIBILITY

MACIEJ P. WOJTKOWSKI

Abstract. A class of left-invariant second order reversible systems with functional parameter is introduced which exhibits the phenomenon of robust integrability: an open and dense subset of the phase space is filled with invariant tori carrying quasi-periodic motions, and this behavior persists under perturbations within the class.

Real-analytic volume preserving systems are found in this class which have positive Lyapunov exponents on an open subset, and the complement filled with invariant tori.

1. Introduction

We study a family of second order dynamical systems on a locally homogeneous Riemannian space M, modeled on a special solvable Lie group. The simplest example is the geodesic flow of a left-invariant metric. Our class generalizes the examples discovered by Butler [Bu1], and Bolsinov and Taimanov [B-T]. In these examples the complete integrability of the geodesic flow in the tangent bundle TM is accompanied by highly non-integrable behavior on an invariant submanifold of codimension $n = dim\ M$. The dynamics there is the suspension of a toral automorphism, and in [B-T] the hyperbolic automorphism is chosen, which leads to an Anosov flow. The presence of an Anosov flow as a subsystem guarantees the positivity of topological entropy.

We show that such a behavior extends to a larger class of left-invariant second order systems. This class is parametrized by a matrix L and a functional parameter F: a smooth vector field in the unit ball of the Euclidean space. We call them $L-F$ *systems*. The crucial property of $L-F$ systems is their J-reversibility, where J is an appropriate involution of the tangent bundle TM. Let us recall that a system is J-*reversible* if the involution J conjugates the forward in time dynamics with the backward in time dynamics.

There is a vast literature devoted to J-reversible systems. The survey paper of Lamb and Roberts [L-R] contains an extensive bibliography.

In particular there is a version of the KAM theory for the J-reversible systems. It goes back to Moser [M], and Sevryuk [S]. In our case we establish robust integrability: it persists under any small perturbation

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as long as we stay in the family of $L-F$ systems. Note that this family is parametrized by an infinite dimensional Banach space of vector fields F. What is notable is that we do not assume volume preservation, the symmetries imposed on the system force the integrability, and the occurrence of a finite absolutely continuous invariant measure. This measure has a density with respect to the Liouville volume which is only C^∞, and typically no real-analytic invariant density exists (Section 8).

The J-reversible KAM theory would give us large subsets of quasi-periodic motions for perturbations which are not left-invariant, as long as they are J-reversible. We were unable to check the non-degeneracy of the unperturbed system required for the application of the KAM theory. However we conjecture that the non-degeneracy does hold for most systems under consideration.

Butler, [Bu2],[Bu3], used the mechanism discovered in [B-T] to obtain C^∞ examples of integrable volume preserving systems with positive metric entropy. In our class we find whole families of real-analytic systems with positive metric entropy and a subset filled with quasi-periodic motions, open but not dense (Section 9).

The phenomenon of robust integrability, accompanied by positive topological entropy occurs already for geodesic flows of linear connections. The generalization of the geodesic flow of the Levi-Civita connection to more general linear connections was discussed in [P-W]. Such a generalization appears naturally in the study of Gaussian thermostats, a class of systems introduced by Hoover [H]. The paper of Gallavotti and Ruelle [G-R] introduces the Gaussian thermostats in the physical context. In particular in our class of systems we find a Gaussian thermostat with the following paradoxical behavior (Section 10). For small kinetic energy the system is asymptotic to an Anosov flow, which has a large codimension in the whole phase space. For larger values of the kinetic energy the system undergoes a drastic change, it becomes integrable: an open and dense subset in the phase space is filled with quasi-periodic motions. The Anosov subsystem is still present, but for a subset of initial conditions of full Lebesgue measure the solutions stay away from that chaotic subsystem, and fill densely invariant tori.

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2. The configuration space

Our configuration space is a locally homogeneous Riemannian space modeled on a special Lie group G. We start its description with the Lie algebra $\mathfrak{g}$. We assume that $dim\ \mathfrak{g} = n+1$ and that $\mathfrak{g}$ contains an n

dimensional abelian ideal $\mathfrak{g}_0$. We choose an arbitrary scalar product in $\mathfrak{g}$, and let b denote a unit vector orthogonal to $\mathfrak{g}_0$. Since the ideal $\mathfrak{g}_0$ is assumed to be abelian the Jacobi identity imposes no conditions on the operator $L : \mathfrak{g}_0 \to \mathfrak{g}_0$, $L = ad_b$. At this stage we place no restrictions on the operator L. Later on we will consider various special cases. We endow the Lie group with the left invariant metric determined by our choice of the scalar product in $\mathfrak{g}$.

The Levi-Civita connection ∇ on the Riemannian manifold G can be expressed as a tensor on $\mathfrak{g}$. It can be calculated directly, which is done in the fundamental paper of Milnor [Mi], where extensive explanations can be found. The formulas read

$$\begin{aligned} &\nabla_b b = 0, \nabla_b \xi = A\xi, \quad \text{for} \quad \xi \in \mathfrak{g}_0, \\ &\nabla_\xi b = -S\xi, , \nabla_\xi \zeta = \langle S\xi, \zeta \rangle b, \quad \text{for} \quad \xi, \zeta \in \mathfrak{g}_0 \end{aligned} \tag{1}$$

where $S = \frac{1}{2}(L + L^*)$ and $A = \frac{1}{2}(L - L^*)$ denote the symmetric and skew-symmetric parts of the operator L.

The Lie algebra has an important automorphism $K : \mathfrak{g} \to \mathfrak{g}$, where $-K$ is equal to the euclidean reflection in $\mathfrak{g}_0$. There are only few Lie algebras with this kind of additional symmetry. It is not difficult to enumerate all of them. Our class of Lie algebras is singled out by the additional property that the invariant subspace of K is a subalgebra.

Since the automorphism K is orthogonal it generates the automorphism $\mathcal{K}$ of the Lie group G which is an isometry. This isometry will play crucial role in our discussion. Let us note that both K and $\mathcal{K}$ are involutive, i.e., $K = K^{-1}, \mathcal{K} = \mathcal{K}^{-1}$.

The Lie group has the following matrix representation, which we will also denote by G. It consists of matrices with the block form

$$\begin{bmatrix} 1 & 0 \\ w & e^{uL} \end{bmatrix}, \quad w \in \mathbb{R}^n, u \in \mathbb{R}.$$

The abelian ideal $\mathfrak{g}_0$ corresponds to the normal abelian subgroup G_0 consisting of the matrices with $u = 0$. We obtain convenient coordinates $(w, u) \in \mathbb{R}^n \times \mathbb{R}$ in the group G, which is the semi-direct product $G_0 \rtimes G_1$, of the abelian additive groups $G_0 = \mathbb{R}^n$ and $G_1 = \mathbb{R}$.

The Lie algebra of our matrix group consists of matrices of the form

$$\begin{bmatrix} 0 & 0 \\ \xi & \eta L \end{bmatrix}, \quad \xi \in \mathbb{R}^n, \eta \in \mathbb{R}.$$

We will be using the linear coordinates $(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}$ in the Lie algebra $\mathfrak{g}$.

Let us choose a lattice Γ_0 in G_0 of rank n, so that $\Gamma_0 \backslash G_0$ is an n-dimensional torus. Our basic configuration space M is the Riemannian manifold $M = \Gamma_0 \backslash G$. The manifold $M = \Gamma_0 \backslash (G_0 \rtimes G_1)$ is canonically diffeomorphic to $(\Gamma_0 \backslash G_0) \times G_1 = \mathbb{T}^n \times \mathbb{R}$. In particular each torus in this product has a canonical affine structure inherited from the abelian

subgroup G_0. This foliation into tori will play an important role. We will refer to the leaves of this foliation as *toral leaves*.

In general the Riemannian manifold M is only locally homogeneous, since the left translations do not factor to the coset space $\Gamma_0\backslash G$. However the left translations by elements from the subgroup G_0 do factor onto M, and these isometries preserve all the toral leaves, acting on them as translations.

The configuration space $M = \Gamma_0\backslash G$ is non-compact. In some cases the discrete subgroup Γ_0 can be enlarged to a discrete subgroup Γ so that the resulting quotient space $N = \Gamma\backslash M$ is compact. It is so when the one parameter subgroup $e^{uL}, u \in \mathbb{R}$, contains an automorphism of $\Gamma_0 \subset \mathbb{R}^n$. In general the necessary condition for the existence of a quotient with finite volume is the unimodularity of the group G, [Mi]. In our case the group G is unimodular if and only if $tr\ L = 0$.

Let us assume for simplicity that $A = e^L$ is an automorphism of the discrete subgroup Γ_0 (and the torus $\Gamma_0\backslash G_0$). The discrete subgroup Γ is generated by Γ_0 and $A \in G$ ($A = e^L$ can be considered as an element of the subgroup G_1: $w = 0, u = 1$.) We will consider such subgroups and the resulting compact locally homogeneous spaces $N = \Gamma\backslash G$.

3. Second order left invariant equations on Lie groups

We will need some general properties of second order equations on Lie groups, which are invariant under left translations.

A *second order equation on a manifold* M is a dynamical system, i.e., a continuous flow $\Psi^t, t \in \mathbb{R}$, on the tangent bundle TM, such that for a trajectory $(x(t), v(t)) = \Psi^t(x(0), v(0)), t \in \mathbb{R}$, where $x(t) \in M$ and $v(t) \in T_{x(t)}M$, we have $\frac{dx}{dt} = v(t)$.

Let further G be a Lie group. The tangent bundle TG of the group is canonically diffeomorphic to the cartesian product $G \times \mathfrak{g}$ by the use of left translations. With this identification the derivative of a left translation on G, as a mapping of $TG = G \times \mathfrak{g}$, is equal to the identity in the second factor.

A second order equation on a Lie group is *left-invariant* if left translations on the tangent bundle $TG = G \times \mathfrak{g}$ commute with the flow. In other words the trajectories of the flow are taken to trajectories by left translations: for any $g \in G$ and any trajectory $(x(t), v(t)) = \Psi^t(x(0), v(0)), t \in \mathbb{R}$, the parametrized curve $(gx(t), v(t))$ in $G \times \mathfrak{g}$ is also a trajectory of the flow.

The structure of a left-invariant second order equation is described in the following

Proposition 3.1. *If a continuous flow Ψ^t on the tangent bundle $TG = G\times\mathfrak{g}$ of a Lie group is a left-invariant second order equation then it is a group extension of a flow on $\mathfrak{g}$, i.e., there is a continuous flow $\psi^t : \mathfrak{g} \to \mathfrak{g}$ and a continuous cocycle over ψ^t with values in G, $h : \mathbb{R} \times \mathfrak{g} \to G$,*

such that

$$\Psi^t(g,v) = (gh(t,v), \psi^t(v)), \quad \textit{for} \ \ (g,v) \in G \times \mathfrak{g}.$$

Proof. Let $\pi_G : TG \to G$ and $\pi_{\mathfrak{g}} : TG \to \mathfrak{g}$ be the projections associated with the identification of TG with $G \times \mathfrak{g}$. We define the flow ψ^t in the Lie algebra by

$$\psi^t(v) = \pi_{\mathfrak{g}}(\Psi^t(g,v)) \ \text{ for any } \ g \in G.$$

The mapping $\psi^t : \mathfrak{g} \to \mathfrak{g}$ is well defined because of left invariance, and hence it must be a flow.

Further let

$$h(t,v) = g^{-1}\pi_G\left(\Psi^t(g,v)\right) \ \text{ for any } \ g \in G.$$

Again since it is well defined the cocyle property can be easily checked. □

The flow on $\mathfrak{g}$ described in Proposition 3.1 will be referred to as the *Euler flow*. This choice of terminology comes from the Euler equation of the rigid body dynamics, see [A], Appendix 2.

It follows from this Proposition 3.1 that if the Euler flow has a periodic trajectory through $v_0 \in \mathfrak{g}$ of period T then the map Ψ^T preserves the set $G \times \{v_0\}$ and it is equal there to a right translation on G.

4. The dynamical systems: the $L - F$ systems

The simplest dynamical system that is of interest to us is the geodesic flow, which we consider in the tangent bundle TM, rather than the cotangent bundle. Using the Riemannian metric we can identify the tangent and cotangent bundles, and the tangent bundle acquires the natural symplectic form ω.

The isometry $\mathcal{K} : G \to G$ projects to $M = \Gamma_0\backslash G$, and we denote it again as $\mathcal{K}$. This isometry is involutive, i.e., $\mathcal{K} = \mathcal{K}^{-1}$.

The derivative $D\mathcal{K} : TM \to TM$ is also involutive and it commutes with the geodesic flow $\Psi^t : TM \to TM, \ \ t \in \mathbb{R}$,

$$D\mathcal{K} \circ \Psi^t = \Psi^t \circ D\mathcal{K}, \quad t \in \mathbb{R}.$$

Let $\tilde{J} : TM \to TM$ be the involution which is identity in the base M (it does not move points in M) and it is equal to minus identity in every tangent space. It is well known that the geodesic flow is $\tilde{J}$-*reversible*, i.e.,

$$\tilde{J} \circ \Psi^t = \Psi^{-t} \circ \tilde{J}, \quad t \in \mathbb{R}.$$

The involutions $\tilde{J}$ and $D\mathcal{K}$ commute, hence their composition is also an involution, and we denote it by $J = \tilde{J} \circ D\mathcal{K} = D\mathcal{K} \circ \tilde{J}$. Since the geodesic flow commutes with $D\mathcal{K}$ it must be also J-reversible.

The J-reversibility is the fundamental self symmetry that is present in our family of dynamical systems on SM.

The first generalization is from the Levi-Civita connection of a left invariant metric on G to a left invariant linear connection $\widehat{\nabla}$ on G. Such a connection differs from the Levi-Civita connection by a tensor B in $\mathfrak{g}$

$$\widehat{\nabla}_X Y - \nabla_X Y = B(X,Y),\ \ X,Y \in \mathfrak{g}.$$

We assume that the connection $\widehat{\nabla}$ has two additional properties. Firstly we require that the parametrization of its geodesics is proportional to the arc length. We will call such connections *para-metric*, since they generalize the concept of *metric connections*, which have isometric parallel transport. A connection is para-metric if and only if $\langle B(X,X), X\rangle = 0$ for every $X \in \mathfrak{g}$. The geodesic flow of a para-metric connection $\widehat{\nabla}$ preserves the unit sphere bundle, and it will be denoted again by $\Psi^t : SM \to SM$. A discussion of geodesic flows of linear connections can be found in [P-W].

Secondly we assume that the symmetric part of the tensor $B = \widehat{\nabla} - \nabla$ is invariant under the isometric involution $K : \mathfrak{g} \to \mathfrak{g}$, i.e., for every $X \in \mathfrak{g}$

$$B(KX, KX) = KB(X,X).$$

If such a property holds we say that the connection is *weakly* $\mathcal{K}$-*invariant*. It follows that for a left-invariant and weakly $\mathcal{K}$-invariant connection the geodesic flow Ψ^t is again J-reversible.

The equations of the geodesic flow can be written as

$$\frac{dx}{dt} = v,\ \ \widehat{\nabla}_v v = 0,$$

where $x(t) \in G$ is a parametrized geodesic.

With the identification of the tangent bundle TG with $G \times \mathfrak{g}$ by left translations we get there the coordinates $(w, u; \xi, \eta) \in \mathbb{R}^{2n+2}$, which were introduced in Section 1. In these coordinates the above involutions are given by

$$\begin{aligned} &D\mathcal{K}(w,u;\xi,\eta) = (-w,u;-\xi,\eta), \tilde{J}(w,u;\xi,\eta) = (w,u;-\xi,-\eta),\\ &J(w,u;\xi,\eta) = (-w,u;\xi,-\eta) \end{aligned}$$

We need to establish the form of a left-invariant, para-metric, weakly $\mathcal{K}$-invariant connection on G. Recall that $b \in \mathfrak{g}$ is a unit vector orthogonal to $\mathfrak{g}_0$. An arbitrary element $X \in \mathfrak{g}$ can be written as $X = \xi + \eta b, \xi \in \mathfrak{g}_0$.

Proposition 4.1. *A connection* $\widehat{\nabla} = \nabla + B$ *on the group* G *is left-invariant, para-metric and weakly* $\mathcal{K}$*-invariant if and only if there is a linear operator* $C : \mathfrak{g}_0 \to \mathfrak{g}_0$ *such that for any* $X = \xi + \eta b, \xi \in \mathfrak{g}_0$*.*

$$B(X,X) = \eta C\xi - \langle C\xi, \xi\rangle b$$

Proof. Without loss of generality we can assume that B is symmetric. We have for every $X = \xi + \eta b, \xi \in \mathfrak{g}_0$,

$$0 \equiv \langle B(X,X),X\rangle = \langle B(\xi,\xi),\xi\rangle + \eta\langle B(\xi,\xi),b\rangle$$
$$+ 2\eta\langle B(\xi,b),\xi\rangle + 2\eta^2\langle B(\xi,b),b\rangle + \eta^2\langle B(b,b),\xi\rangle + \eta^3\langle B(b,b),b\rangle.$$

Since this cubic polynomial in η must vanish we get for every $\xi \in \mathfrak{g}_0$

$$\langle B(\xi,\xi),b\rangle + 2\langle B(\xi,b),\xi\rangle = 0, 2\langle B(\xi,b),b\rangle + \langle B(b,b),\xi\rangle = 0,$$
$$\langle B(\xi,\xi),\xi\rangle = 0, \langle B(b,b),b\rangle = 0.$$

Since the connection is weakly $\mathcal{K}$-invariant we must have $KB(b,b) = B(b,b)$ and $KB(\xi,\xi) = B(\xi,\xi)$, and so we obtain further that $B(b,b) = 0$ and $B(\xi,\xi) \perp \mathfrak{g}_0$. It follows readily that

$$\langle B(\xi,b),b\rangle = 0,$$
$$B(\xi,\xi) = -2\langle B(\xi,b),\xi\rangle b.$$

Putting $C\xi = 2B(\xi,b)$ we get the desired formula. □

Using Proposition 4.1 and the formulas (1) we obtain by direct calculations the equations of the geodesic flow for our special connections.

Theorem 4.2. *For any para-metric left-invariant and weakly $\mathcal{K}$-invariant connection, the geodesic equations in the coordinates $(w,u;\xi,\eta)$ in the tangent bundle TG are*

$$\frac{d\xi}{dt} = \eta F(\xi), \quad \frac{d\eta}{dt} = -\langle F(\xi),\xi\rangle$$
$$\frac{dw}{dt} = e^{uL}\xi, \quad \frac{du}{dt} = \eta. \tag{2}$$

where $F(\xi) = L^*\xi - C\xi$, and the matrix C depends only on the tensor $B = \widehat{\nabla} - \nabla$, namely for $X = \xi + \eta b, \xi \in \mathfrak{g}_0$ we have $B(X,X) = \eta C\xi - \langle C\xi,\xi\rangle b$. □

Note that in the last Theorem any matrix C can occur with an appropriate choice of the connection $\widehat{\nabla}$.

The equations (2) factor to the Lie algebra $\mathfrak{g}$ as *the Euler equations*, the first line of (2). Let us recall that the Euler equations are obtained by left translations of velocities along geodesics. It follows from the left invariance of the connection that the resulting curves in $\mathfrak{g}$ must satisfy the Euler equation, see [A], Appendix 2.

Our final generalization is to replace the linear vector field $F(\xi) = F\xi, \xi \in \mathfrak{g}_0 = \mathbb{R}^n$, in (2) by a general (non-linear) vector field $F : \mathfrak{g}_0 \to \mathfrak{g}_0$. Such a dynamical system will be called an *$L - F$ system.* In the special case of a left invariant para-metric connection, when the vector field F is linear, we will call it a *quadratic $L - F$ system.*

To summarize: the operator $L : \mathfrak{g}_0 \to \mathfrak{g}_0$ in the euclidean space $\mathfrak{g}_0$ determines the Lie group G with a chosen left invariant metric. The

vector field F determines then the second order equations (2) in TG which are preserved by any left translation of G, and hence project naturally to TM, or SM. Note that to define an $L-F$ system on SM it is enough to have the vector field F defined in the closed unit ball of $\mathfrak{g}_0$.

The crucial property of general $L-F$ systems on SM (or TM) is their J-reversibility.

Actually the $L-F$ systems can be characterized as smooth second order left invariant equations on SM which are J-reversible.

Proposition 4.3. *Any second order left-invariant equations on SM which are also J-reversible define an $L-F$ system.*

Proof. By Proposition 3.1 the equations factor from SM to the unit sphere $\mathbb{S}^n = \{(\xi,\eta) \in \mathfrak{g} | \xi^2+\eta^2 = 1\}$. The resulting Euler system is also J-reversible, where $J(\xi,\eta) = (\xi,-\eta)$.

Using the coordinates $(\xi_2,\dots,\xi_n,\eta)$ on the unit sphere in the neighborhood of the equator $\{\eta = 0\}$ we consider the smooth function

$$U(\xi,\eta) = \frac{d\xi}{dt}.$$

By the J-reversibility we obtain that the function U is odd in the η variable, i.e., $U(\xi,-\eta) = -U(\xi,\eta)$. It follows that if U is a smooth function on the unit sphere then the function

$$F(\xi) = \frac{1}{\sqrt{1-\xi^2}} U(\xi,\sqrt{1-\xi^2})$$

is well defined and smooth in the closed unit ball in $\mathfrak{g}_0$, and $U(\xi,\eta) = \eta F(\xi)$. □

5. Periodic solutions in J-reversible systems

For J-reversible systems there is a very convenient way of searching for periodic solutions. It goes so far back that it is by now a part of the mathematical folklore. It was formulated explicitly by DeVogelaere [DeV], and Devaney [D]. We do it again for the convenience of the reader.

Let us consider the subset $\mathcal{F}$ of fixed points of the involution J, $\mathcal{F} = \{p|\ J(p) = p\}$.

Theorem 5.1. *Any trajectory of a J-reversible flow Ψ^t which visits $\mathcal{F}$ twice must be periodic. Moreover such a trajectory is invariant under J with the reversal of time.*

Proof. Let $p_0 \in \mathcal{F}$ be such that there is $t_0 > 0$ with $\Psi^{t_0}(p_0) \in \mathcal{F}$. We have

$$\Psi^{t_0}(p_0) = J\left(\Psi^{t_0}(p_0)\right) = \Psi^{-t_0} J(p_0) = \Psi^{-t_0}(p_0).$$

Hence the trajectory $p(t) = \Psi^t(p_0)$ is periodic with the period $T = 2t_0$. Moreover

$$Jp(t) = J\Psi^t(p_0) = \Psi^{-t}(Jp_0) = \Psi^{-t}(p_0) = p(-t).$$

□

The minimal period of a trajectory of p_0 in the above proof is $T = 2t_0$ if and only if $t_0 > 0$ is the time of the first return of p_0 to the set $\mathcal{F}$ of fixed points of J.

We apply this principle not to the full $L - F$ system but only to its factor, the Euler equations

$$\frac{d\xi}{dt} = \eta F(\xi), \ \frac{d\eta}{dt} = -\langle F(\xi), \xi\rangle, \tag{3}$$

The involution J descends naturally to the phase space of (3) $(\xi, \eta) \in \mathfrak{g}$, and we denote it again by J, $J(\xi, \eta) = (\xi, -\eta)$. Clearly the Euler equation is J-reversible. The set of fixed points of J is equal to $\mathcal{F} = \mathfrak{g}_0 = \{(\xi, \eta) | \eta = 0\}$.

Let us consider the open unit ball $B \subset \mathfrak{g}_0$, with the boundary, the unit sphere, $S = \partial B$.

Definition 5.1. For a smooth vector field $F = F(\zeta)$ defined on the closed unit ball in $\mathfrak{g}_0$, we say that a point ζ_0 in the open unit ball B is *escaping* if the integral curve $\zeta = \zeta(s)$ of F through ζ_0 is defined in a finite closed interval $[s_-, s_+] \ni 0$, $\zeta(0) = \zeta_0$, $\zeta(s) \in B$ for $s \in (s_-, s_+)$, the endpoints $\zeta(s_\pm)$ belong to the unit sphere $S = \partial B$, and the vector field F is transversal to the unit sphere S at the endpoints of the integral curve, i.e.,

$$\langle F(\zeta(s_-)), \zeta(s_-)\rangle < 0, \ \langle F(\zeta(s_+)), \zeta(s_+)\rangle > 0. \tag{4}$$

The integral curve through an escaping point is called an *escaping trajectory*.

Let us note that for a given smooth vector field F the set of escaping points in the unit ball is open.

We have the following crucial

Theorem 5.2. *For any escaping trajectory $\zeta(s), s \in [s_-, s_+]$ of the vector field $f = F(\zeta), \zeta \in \mathfrak{g}_0$, there are periodic functions $\eta(t)$ and $u(t)$ with the period $T = 2t_0$ such that $\eta(0) = \eta(t_0) = 0, u(0) = s_-, u(t_0) = s_+, s_- \le u(t) \le s_+$, and $(\xi(t), \eta(t))$ is a T-periodic solution of the Euler equation (3), where $\xi(t) = \zeta(u(t))$. Moreover $\eta(t)$ is an odd function $\eta(-t) = -\eta(t), t \in \mathbb{R}$, and $u(t)$ is an even function*

$$u(t) = s_- + \int_0^t \eta(s) ds.$$

Proof. We restrict the Euler equation (3) to the upper half of the unit sphere in $\mathfrak{g}$, i.e., to the subset $\{(\zeta, \eta) | \zeta^2 + \eta^2 = 1, \eta > 0\}$. In this

submanifold we can use ζ as coordinates, and we introduce there the time change $\frac{ds}{dt} = \eta$. After this time change the Euler equation becomes the following system

$$\frac{d\zeta}{ds} = F(\zeta). \tag{5}$$

Hence the trajectory $\zeta(s), s_- < u < s_+$, of this vector field in the open ball $B \subset \mathfrak{g}_0$ gives rise to the trajectory $(\xi(t), \eta(t)), t_- < t < t_+$, of the Euler equation in the upper half of the unit sphere in $\mathfrak{g}$, with $\lim_{t\to t_-} \eta(t) = \lim_{t\to t_+} \eta(t) = 0$. It may happen (and it does) that $t_\pm = \pm\infty$. This is excluded in our case by the condition (4) which gives us $\lim_{t\to t_-} \frac{d\eta}{dt} > 0, \lim_{t\to t_+} \frac{d\eta}{dt} < 0$. Hence we have a finite time interval $[t_-, t_+]$ and we can shift it to the time interval $[0, t_0], t_0 = t_+ - t_-$.

By Theorem 5.1 this trajectory extends to the $T = 2t_0$ periodic solution of the Euler equation $(\xi(t), \eta(t)), t \in \mathbb{R}$. Putting $u(t) = s_- + \int_0^t \eta(s)ds$ we have $\xi(t) = \zeta(u(t))$ for $0 \le t \le t_0$. Since the periodic solution is invariant under J with the reversal of time, we obtain that $\xi(-t) = \xi(t), \eta(-t) = -\eta(t)$. It follows that $\int_{-t_0}^{t_0} \eta(s)ds = 0$, and consequently $u(t)$ is a T periodic function $0 \le u(t) \le u_0$. Hence we have also $\xi(t) = \zeta(u(t))$ for any $t \in \mathbb{R}$. $\square$

In simple terms what this proof reveals is that an integral curve of F is up to a time change also a trajectory of the respective Euler flow in the upper semi-sphere. For escaping trajectories of F the solution of the Euler equation crosses transversally the "equator" $\{\eta = 0\}$ and in the lower semi-sphere it follows the same integral curve of F but in the reversed direction. Hence it must be a periodic trajectory of the Euler flow.

In the case of a quadratic $L - F$ system we get the following

Corollary 5.3. *If the linear vector field $F(\xi) = F\xi$ has eigenvalues both with positive and negative real parts, then for the Euler equation (3) the periodic trajectories fill an open and dense subset of $\mathbb{S}^n \subset \mathfrak{g}$.*

If all the eigenvalues of F are on the imaginary axis then for the Euler equation (3) an open and dense subset of $\mathbb{S}^n \subset \mathfrak{g}$ is filled with trajectories which are either periodic or quasi-periodic after an appropriate smooth time change.

Proof. Let us first note that for linear vector field F the condition (4) is satisfied on an open and dense subset of the unit sphere $S \subset \mathfrak{g}_0$, unless $F^* = -F$.

If the matrix F has eigenvalues with both positive and negative real parts then the set of escaping points in the unit ball $B \subset \mathfrak{g}_0$ is not only open but also dense in the unit ball. That is so because the instability as $t \to \pm\infty$ forces typical solutions to reach the boundary unit sphere at some finite time both in the future and in the past. By Theorem 5.2 they give rise to the periodic solutions of the Euler equations.

If the matrix F has only purely imaginary eigenvalues, and there is no resonance, then the linear system (5) has only quasi-periodic solutions. In general there is an open set of trajectories which stay in the unit ball for all times, and an open set of escaping trajectories. The union of these open sets is dense in the unit sphere. The only exception is the case of the skew-symmetric matrix $F^* = -F$, when there are no escaping trajectories. Note that while the solution of (5) is quasi-periodic the solution of the Euler equation need not be quasi-periodic since a time change is involved.

In the resonant case we have instability for $t \to \pm\infty$, and hence there is a dense subset of escaping trajectories, both in the cases of purely imaginary nonzero eigenvalues and of the zero eigenvalue. By Theorem 5.2 we get an open and dense subset of the unit sphere $\mathbb{S}^n \subset \mathfrak{g}$ filled with periodic solutions of the Euler equations. □

6. Robust integrability of $L - F$ systems

The notion of integrability of a dynamical system has a long history and several different versions. In hamiltonian dynamics the main concept of integrability is associated with the Liouville-Arnold theorem, where families of invariant tori carrying quasi-periodic motions appear. The issues involved in the general definition of integrability in hamiltonian dynamics were explored by Bogoyavlensky, [Bo1],[Bo2], and Fasso [F].

We study a family of reversible, non-hamiltonian systems in which quasi-periodic motions occur robustly. We introduce the following working definition

Definition 6.1. We call a dynamical system *semi-integrable* if an open subset U of the phase space is filled with invariant tori carrying the quasi-periodic motions. If U is also dense in the phase space then the dynamical system will be called *integrable*.

It turns out that any periodic solution of the Euler equation constructed in Theorem 5.2 gives rise to quasi-periodic solutions of the $L - F$ system. This is the contents of the following theorems.

Theorem 6.1. *If a vector field F in $\mathfrak{g}_0$ has escaping points in the unit ball then the $L - F$ system on SM is semi-integrable.*

If the vector field F has a dense set of escaping points in the unit ball then the $L - F$ system on SM is integrable.

For quadratic $L - F$ system we can formulate an effective criterion of integrablity. Moreover the integrablity persists under small perturbations in the space of $L - F$ systems.

Theorem 6.2. *If for a linear vector field $F = F\xi, \xi \in \mathfrak{g}_0$ the matrix F has eigenvalues with both positive and negative real parts then the*

quadratic $L-F$ system on TM is integrable. Moreover the invariant tori are common level sets of real-analytic first integrals.

Any small perturbation of such a quadratic integrable $L-F$ system in the space of all $L-F$ systems must be integrable.

The last theorem covers the examples of geodesic flows in Butler,[Bu1], and Bolsinov and Taimanov, [B-T].

Corollary 6.3. *If the operator $L : \mathfrak{g}_0 \to \mathfrak{g}_0$ has eigenvalues with both positive and negative real parts then for the left-invariant metric on M the geodesic flow on TM is integrable.*

Moreover the $(n+1)$-dimensional invariant tori are common level sets of real-analytic first integrals in involution.

It seems that within the class of quadratic integrable systems, for an open dense subset of operators F the system is non-degenerate, in the sense of having a rich family of frequencies on SM. The reversible version of the KAM theory, developed by Moser, [M], and Sevryuk, [S], would be applicable upon the establishment of the non-degeneracy. The allowed perturbations would be J-reversible second order equations, but not necessarily left-invariant.

The calculations required to establish the non-degeneracy are cumbersome, and we did not find a satisfactory way to do it.

We will give a joint proof of Theorems 6.1 and 6.2.

Proof. We will establish that the periodic solution of the Euler equation in $\mathfrak{g}$ constructed in Theorem 5.2 is covered by invariant tori in TM carrying quasi-periodic solutions. To that end, given the T-periodic solution $v(t) \in \mathfrak{g}, v(t) = (\xi(t), \eta(t))$ of Theorem 5.2, $T = 2t_0$, let us integrate the equations (2) of the $L-F$ system. We get

$$u(t) = a + a_0(t),\ \ a_0(t) = \int_0^t \eta(s)ds,$$
$$w(t) = b + b_0(t),\ \ b_0(t) = \int_0^t e^{u(s)L}\xi(s)ds.$$

Since the function η is odd and periodic we obtain that $a_0(t)$ is also a periodic even function and assumes values between $-u_0$ and u_0, where $u_0 = \int_0^{t_0} \eta(s)ds$. Note that although $b_0(t)$ is not periodic, its time derivative is T-periodic.

Fixing the value $a = u(0)$ we obtain an invariant torus. Indeed consider the flow Ψ^t defined by the $L-F$ system (3), which is a group extension flow by Proposition 3.1. The mapping Ψ^T takes $M \times \{v(0)\}$ into itself and it acts there as a right translation. Since $u(t)$ is periodic the translation is by an element from G_0, and hence it preserves all the toral leaves. For $a \in \mathbb{R}$ and $v \in \mathfrak{g}$ denote by $\mathcal{L}(a,v)$ the toral leaf $\mathcal{L}(a,v) = \{(w,u;\xi,\eta) \in SM | w \ mod\ \Gamma_0, u = a, (\xi,\eta) = v\}$. Since Ψ^T takes $\mathcal{L}(a, v(0))$ into itself, and it acts there by the translation

by $c_0 = \int_0^{2t_0} e^{u(s)L}\xi(s)ds$, then the union of the n-dimensional tori $\Psi^t\mathcal{L}(a, v(0)), t \in \mathbb{R}$, is an $(n+1)$-dimensional torus carrying the suspension flow of the toral translation, i.e., a quasi-periodic flow.

In the case of a linear vector field $F(\xi) = F\xi$, for some constant matrix F, the real analytic first integrals are given by the vector valued function

$$\Phi(w, u; \xi, \eta) = e^{-uF}\xi.$$

These first integrals are functionally independent, hence their common level sets must be in general unions of submanifolds of dimension $n+1$. Clearly the toral leaves $\mathcal{L}(a, v)$ belong to the level sets, and so do the $n+1$-dimensional tori constructed above. Hence in general the level sets must be unions of the $(n+1)$-dimensional invariant tori. □

The geodesic flow on TM is both an $L-F$ system and hamiltonian but it is an exception. The $L-F$ systems are rarely hamiltonian.

To establish the Corollary 6.3 we consider more generally hamiltonian systems on T^*M with G_0 symmetry. If the hamiltonian function $H : T^*M \to \mathbb{R}$ is invariant under the action of the torus G_0/Γ_0 then we get immediately n first integrals in involution. Indeed, this action in the canonical variables $(w, u; p_w, p_u)$, associated with the variables (w, u) in M, amounts to translations in w with the other variables fixed. It follows that the hamiltonian H does not depend on the variables w and so the n momenta p_w are first integrals of our hamiltonian system. Together with H itself we have $n+1$ first integrals in involution, and if H is functionally independent of p_w then the Arnold-Liouville Theorem is applicable. In particular we obtain invariant tori on compact level sets of H.

Let us identify the cotangent bundle T^*M with the tangent bundle TM using the left invariant metric on M. This identification is the Legendre transform associated with the geodesic flow. Since the lagrangian $\mathcal{L}$ for the geodesic flow is equal to

$$\mathcal{L}(w, u; \dot{w}, \dot{u}) = \frac{1}{2}\left(\langle e^{-uL}\dot{w}, e^{-uL}\dot{w}\rangle + \dot{u}^2\right),$$

we get immediately that

$$p_w = \frac{\partial}{\partial \dot{w}}\mathcal{L} = e^{-uL^*}e^{-uL}\dot{w} = e^{-uL^*}\xi.$$

For the geodesic flow these are the same first integrals as those in Theorem 6.2.

Let us finally consider more special hamiltonians, which are invariant under the action of the full group G. They can be described by functions $\widetilde{H} : \mathfrak{g} \to \mathbb{R}$, $\widetilde{H} = \widetilde{H}(\xi, \eta)$ and $H = \widetilde{H}(e^{uL^*}p_w, p_u)$. It is instructive

to compare the respective hamiltonian equations in the $(w, u; \xi, \eta)$ variables in TM with the equations of an $L-F$ system. We have

$$\frac{d\xi}{dt} = \frac{\partial \widetilde{H}}{\partial \eta} L^*\xi, \quad \frac{d\eta}{dt} = -\langle \frac{\partial \widetilde{H}}{\partial \xi}, L^*\xi \rangle$$
$$\frac{dw}{dt} = e^{uL}\frac{\partial \widetilde{H}}{\partial \xi}, \quad \frac{du}{dt} = \frac{\partial \widetilde{H}}{\partial \eta}.$$

It transpires that also in the hamiltonian case if the function $\widetilde{H}$ has compact level sets then the Euler equation in $\mathfrak{g}$ has an open and dense set of periodic trajectories.

7. Compact configuration spaces

In this section we consider the additional properties of an $L-F$ system on the compact phase space, namely the unit sphere bundle SN. where Γ is a cocompact lattice in G and $N = \Gamma\backslash G$, as discussed in Section 1.

Let us note first that in this case the group must be unimodular, and the operator L has zero trace. Hence either L has eigenvalues with both positive and negative real parts, or all of its eigenvalues are on the imaginary axis.

The integrability and semi-integrability are passed from the noncompact phase space SM to the compact phase space SN without further assumptions. What is new is the appearance of hyperbolic behavior made possible by the recurrence in the compact phase space. For any quadratic $L-F$ system the submanifold $\mathcal{A} \subset SM$ given by the equations $\xi = 0, \eta = 1$ is diffeomorphic to M and it carries trajectories escaping to infinity both in the future and in the past. The projection of $\mathcal{A}$ to the compact phase space SN is diffeomorphic to N and the $L-F$ flow is the suspension of the toral automorphism e^L. If the toral automorphism has no eigenvalues on the unit circle then it is an Anosov diffeomorphism, and its suspension is an Anosov flow. If it has only some eigenvalues outside the unit circle then we get partially hyperbolic flows as suspensions, [K-H]. Such flows in the compact phase space SN while integrable have positive topological entropy. The existence of such flows was the discovery of Bolsinov and Taimanov, [B-T].

More generally let us assume that the automorphism e^L has eigenvalues outside of the unit circle. Then for an arbitrary left-invariant second order equations on SN (not necessarily an $L-F$ system) we have the following

Proposition 7.1. *If the variable η has positive lower (upper) time average on some trajectory then this trajectory has negative and positive lower (upper) Lyapunov exponents. The dimension of the respective stable and unstable subspaces is equal to the dimension of the stable and unstable subspaces of the automorphism e^L.*

Proof. The flow Ψ^t has the structure of a group extension flow over the Euler flow as described in Proposition 3.1. We fix a trajectory $v(t) = (\xi_0(t), \eta_0(t))$ of the Euler flow. There is a unique parametrized curve $(w_0(t), u_0(t)) \in G, w_0(0) = 0, u_0(0) = 0$, such that for every $(w, u) \in G$

$$\Psi^t(w, u; v(0)) = \left(w + e^{uL} w_0(t), u + u_0(t); v(t)\right).$$

Hence the flow takes the n- dimensional toral leaf $\mathcal{L}(u, v(0))$ onto the toral leaf $\mathcal{L}(u + u_0, v(t))$. We have the coordinates w for both toral leaves. In these coordinates the restriction of the flow is defined by $w \mapsto w + e^{uL} w_0(t)$. Differentiating this mapping with respect to w we obtain that $D\Psi^t$ restricted to the n-dimensional tangent subspace of the toral leaf is given by the identity operator in the coordinates w. However we measure the tangent vectors using the left-invariant metric and that means

$$||dw||^2 = \left(e^{-(u+u_0)L} dw\right)^2.$$

Hence we get the exponential growth (decay) as long as u_0 has linear growth (decay). Since $\frac{du_0}{dt} = \eta_0$, the assumption of the positive time average of η_0 leads to the exponential growth (decay) of $||dw||^2$ in the unstable (stable) subspace of the operator L. □

We have not chosen any invariant measure so the Lyapunov exponents need to be understood as the upper (or lower) limits. It follows from the proof that assuming negative lower (upper) time average we will arrive at the same conclusion, with the reversal of the number of positive and negative Lyapunov exponents.

The phenomenon described in Proposition 7.1 appears explicitly in the work of Butler and Paternain on special magnetic flows on SOL, [Bu-P].

Let us consider first integrals of a quadratic $L - F$ flow Ψ^t on the compact phase space SN. The real-analytic first integrals of Theorem 6.1 can be considered as real analytic multi-valued first integrals. We will establish that for a residual subset of integrable quadratic $L - F$ systems on SN there are no single valued real-analytic first integrals constant on the toral leaves. If an integrable system is non-degenerate (i.e., it has a rich family of frequencies for the quasi-periodic motions on the tori) then by necessity any continuous first integral must be constant on the toral leaves. Since we have not established the non-degeneracy we need to include the constancy into the assumptions.

Let us recall that by Theorem 6.1 quadratic $L - F$ flows on SN for which the matrix F has eigenvalues with positive and negative real parts are integrable, and they form an open subset in the space of all quadratic $L - F$ flows. The complement of this subset has an interior filled with flows asymptotic to the suspension of the toral automorphism e^L.

Proposition 7.2. *For a residual subset of integrable quadratic $L-F$ flows on SN there are no single valued real-analytic first integrals constant on the n-dimensional toral leaves, i.e., functions of $(u;\xi,\eta)$.*

Proof. We consider only diagonalizable operators F, with all different eigenvalues $(\lambda_1,\lambda_2,\dots,\lambda_n)$, at least some of them with positive and some of them with negative real parts. This gives us an open and dense subset of integrable quadratic $L-F$ systems. If there is a real-analytic first integral R for such a system on SN then it can be lifted to the first integral of the system on SM, which we denote also by R. Now R can be considered as a function of $(u;\xi,\eta)$. We restrict our attention to the unit ball B in $\mathfrak{g}_0$ (the "ξ-space"), and we put $\eta=\sqrt{1-\xi^2}$. This gives us a real-analytic function

$\widetilde{R}(u;\xi),(u;\xi)\in\mathbb{R}\times B$, which is also a first integral of the system.

We further consider the time change $\frac{ds}{dt}=\eta$ in our domain $\mathbb{R}\times B$ The equations of the system in the variables $(u;\xi)$ are

$$\frac{du}{ds}=1,\ \frac{d\xi}{ds}=F\xi \tag{6}$$

The function $\widetilde{R}$ is a first integral of (6) so that its time derivative is identically zero

$$\frac{\partial\widetilde{R}}{\partial u}+\frac{\partial\widetilde{R}}{\partial\xi}F\xi\equiv 0.$$

It follows that for the real analytic function $P(u;\zeta)=\widetilde{R}(u;e^{uF}\zeta)$ defined in the vicinity of the origin we have

$$\frac{\partial P}{\partial u}=\frac{\partial\widetilde{R}}{\partial u}+\frac{\partial\widetilde{R}}{\partial\xi}Fe^{uF}\zeta\equiv 0.$$

Hence P is the function of ζ alone defined in the vicinity of the origin in $\mathbb{R}^n$ such that $\widetilde{R}(u;\xi)=P(e^{-uF}\xi)$. Now we use the fact that the function $\widetilde{R}(u;\xi)=P(e^{-uF}\xi)$ must be actually a periodic function of u with the period 1, since $\widetilde{R}$ is lifted from SN. Hence

$$P(e^{-F}\xi)=P(\xi),\quad\text{for all}\quad \xi\in\mathbb{R}^n. \tag{7}$$

It goes back to Poincare that such a relation is impossible if the eigenvalues of F are non-resonant. More precisely, if there is no integer vector $(k_1,k_2,\dots,k_n)$, with non-negative entries, such that

$k_1\lambda_1+\dots+k_n\lambda_n=0$ then the relation (7) can hold only for a constant function.

The family of $L-F$ systems with non-resonant eigenvalues form a residual subset of the open set of integrable quadratic $L-F$ systems introduced at the beginning of the proof. $\square$

8. Invariant measures

In this section we discuss smooth invariant measures for the $L-F$ systems, for the non-compact SM and compact SN phase spaces. First of all we consider the Lebesgue measure $dw\,du$ in G, which is the right-invariant Haar measure. It is also invariant under the left translations by elements from Γ_0. So it projects into a σ-finite measure μ on $M = \Gamma_0\backslash G$. The product of μ by the standard Lebesgue measure in $\mathbb{S}^n \subset \mathfrak{g}$ (or $\mathfrak{g}$) will be referred to as the Lebesgue measure in $SM = M \times \mathbb{S}^n$ (or TM) and denoted by ν. Once the Lebesgue measure ν is chosen we call a function $\rho \geq 0$ an *invariant density* of a dynamical system if the measure $\rho\nu$ is preserved by the dynamical system.

Proposition 8.1. *If the vector field F has constant divergence in the unit ball of $\mathfrak{g}_0$ then $\rho = exp(-u\ div\ F)$ is an invariant density for the $L-F$ system on SM.*

Proof. Since the $L-F$ system defined in $TM = M \times \mathfrak{g}$ preserves all the sphere bundles it suffices to check the invariance of the measure in TM. It can be easily seen that the divergence of the $L-F$ system in TM is equal to $\eta\ div\ F$. Since

$$\frac{d}{dt}\rho = -\eta\ div\ F,$$

the claim is proven. □

In the case of a compact phase space SN the group G must be unimodular, i.e., $tr\ L = 0$. For unimodular Lie groups the left Haar measure is equal to the right Haar measure, and the Lebesgue measures μ and ν project as finite measures to N and SN, respectively. We will denote the resulting measures again as μ and ν.

It follows from the above Proposition that if $div\ F = 0$ then the $L-F$ system preserves the Lebesgue measure ν in SN. Moreover the Euler flow of the $L-F$ system preserves the Lebesgue measure on $\mathbb{S}^n \subset \mathfrak{g}$.

Conversely, if an $L-F$ system has a (real-analytic) invariant density ρ in SN then the Euler flow in $\mathbb{S}^n \subset \mathfrak{g}$ has a (real-analytic) invariant density. Indeed, we obtain the density of the projected measure by integrating out the w, u coordinates. It preserves the smoothness, and the real-analyticity of the density.

For the quadratic $L-F$ systems the Euler flow cannot have a smooth invariant density which is positive at the fixed point $\xi = 0$, unless $div\ F = 0$. However there may be smooth, or even real-analytic, invariant densities which vanish at the fixed point, even if $div\ F \neq 0$.

Proposition 8.2. *For a residual subset in the space of quadratic $L-F$ systems in SN there are no real-analytic invariant densities.*

For the open family of integrable quadratic $L-F$ systems in SN with the matrix F having all different real eigenvalues, there are C^∞

invariant densities. Moreover there is a dense subset in this family with real-analytic invariant densities.

Proof. It follows from the above considerations that if an $L-F$ system in SN has a real analytic invariant density then the field F has a real analytic invariant density $\rho = \rho(\xi)$ in the unit ball in $\mathfrak{g}_0$. Hence we have

$$div\,(\rho F) = 0.$$

For a diagonalizable matrix F we can analyze the last equation in the coordinates in which F is diagonal. This may require a linear complex change of coordinates, but that is not a problem in calculations involving Taylor expansions. Assuming that F is already diagonal with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ we get

$$\sum_{i=1}^{n} \lambda_i \xi_i \frac{\partial \rho}{\partial \xi_i} = -trF\rho \tag{8}$$

This relation can hold for a function real-analytic in the neighborhood of the origin $\xi = 0$ if and only if there are monomials which satisfy it. For $\rho(\xi) = \prod_{i=1}^{n} \xi^{r_i}$ we get substituting into (8)

$$\sum_{i=1}^{n} \lambda_i (r_i + 1) = 0. \tag{9}$$

The set of matrices F for which (9) does not hold for any vector $(r_1, r_2, \dots, r_n)$ with natural entries is a residual subset in the space of all matrices. The first part of the Theorem is proven.

To prove the second part consider the set of matrices with all different real eigenvalues, at least one positive and one negative. It is an open set of matrices. Its subset where the eigenvalues satisfy (9) for some vector $(r_1, r_2, \dots, r_n)$ with natural even entries is dense. For such systems the function $\rho(\xi) = \prod_{i=1}^{n} \xi^{r_i} \geq 0$ is an invariant density of the Euler flow, and also the full $L-F$ system in SN.

To get a C^∞ invariant density we choose a vector $(r_1, r_2, \dots, r_n)$ with positive entries satisfying (9) and consider the function $\widetilde{\rho}(\xi) = \prod_{i=1}^{n} |\xi|^{r_i} \geq 0$. The function satisfies (8) in the open dense subset where it is positive, but it is not in general a smooth function. We consider the first integral of the Euler flow $f(\xi) = \prod_{i=1}^{n} |\xi|^{r_i+1}$ and we get a C^∞ invariant density $\rho(\xi) = exp(-f(\xi)^{-1})\widetilde{\rho}(\xi)$. □

The C^∞ invariant densities in the proof are similar to the C^∞ first integrals of Butler, [Bu1].

9. Coexistence of integrablility and positive metric entropy

The results of Sections 6 and 7 may be used to find real-analytic volume preserving $L-F$ flows with both quasi-periodic motions in an open

subset and positive metric entropy. Such examples were constructed by Butler in the C^∞ class, [Bu2],[Bu3]. The recent survey of Chen, Hu and Pesin, [C-H-P] describes other kinds of coexistence phenomenae.

Let F have purely imaginary all different eigenvalues, but $F^* \neq -F$. For such a linear vector field the open unit ball is not invariant, hence there is an open subset of escaping points. More precisely the whole open unit ball is the closure of the union of two open subsets V_e and V_b. The open subset V_e contains only escaping points, and the open subset V_b contains points whose trajectories have compact closures in the unit ball. Actually these compact closures are invariant tori of the linear system defined by the matrix F. Note also that the linear vector field has zero divergence, so the $L-F$ flow preserves the Lebesgue measure.

These considerations apply already for $n = 2$. If F has purely imaginary eigenvalues, but $F^* \neq -F$, then the open unit disk contains an open ellipse filled with elliptical integral curves, The rest of the open unit disk is filled with escaping trajectories.

The open set of escaping trajectories V_e gives rise by Theorem 6.1 to semi-integrability of the $L - F$ flow. If the matrix L has eigenvalues with positive (and negative) real parts then the open set of bounded trajectories V_b leads by Proposition 7.1 to positive Lyapunov exponents. By Pesin formula, [K-H], we obtain also positive metric entropy with respect to the invariant Lebesgue measure in SN. We get

Theorem 9.1. *There are volume-preserving quadratic $L - F$ systems on SN which are semi-integrable and have positive metric entropy.*

Let us note that these properties will hold also for small divergence free perturbations of F as long as we guarantee that the perturbed vector field in $\mathfrak{g}$ still has a positive Lebesgue measure of quasi-periodic motions near the origin $\xi = 0$. This can be achieved for instance when the linear vector field F and the perturbation are hamiltionian and satisfy the non-degeneracy conditions of the KAM theory, [A], Appendix 8.

Further let us consider the modified linear vector field $F_1 = F - \epsilon I$. We obtain for $\epsilon > 0$ the asymptotic stability of the origin $\xi = 0$. At the same time for small ϵ the restriction of the vector field F_1 to the unit ball still has an open set of escaping trajectories.

Theorem 9.2. *There is an open set of quadratic $L - F$ systems on SN which are semi-integrable and the interior of the complement to the set of invariant tori is filled with orbits which are asymptotic as $t \to +\infty$ $(-\infty)$ to the attractor (repellor) carrying the suspension of the toral automorphism e^L (e^{-L}).*

We conjecture that the following is true.

Conjecture. *There are real-analytic divergence free vector fields in the closed unit ball with an open and dense subset filled with escaping*

trajectories, and with the complement of positive Lebesgue measure, filled with trajectories defined for all times and which are contained in the interior of the unit ball.

In the Arnold diffusion scenario a generic hamiltonian perturbation of an integrable hamiltonian system has a nowhere dense subset of invariant tori of positive Lebesgue measure. In the complement almost all orbits move away unboundedly both in the future and in the past. However this scenario was not so far rigorously established in any real-analytic examples.

If this conjecture holds then the respective real-analytic $L-F$ system would be integrable and of positive metric entropy.

10. Gaussian thermostats

A Gaussian thermostat is defined by a Riemannian metric on a manifold M and a vector field E. The trajectories of the Gaussian thermostat satisfy the following ordinary differential equations in the tangent bundle TM,[P-W].

$$\frac{d}{dt}x = v, \nabla_v v = E - \frac{\langle E, v\rangle}{\langle v, v\rangle} v, \tag{10}$$

where $x = x(t) \in M$ is a parametrized curve in M.

By the force of these equations the "kinetic energy" v^2 is constant. Fixing the value of this constant $k = v^2$, and introducing the auxiliary vector field $F_k = \frac{1}{k}E$ we can rewrite the equations (10) as

$$\frac{d}{dt}x = v, \nabla_v v = v^2 F_k - \langle F_k, v\rangle v. \tag{11}$$

It was observed in [P-W] that the equations (11) describe geodesics of a special metric connection $\widetilde{\nabla}$ defined by the field F_k

$$\widetilde{\nabla}_X Y = \nabla_X Y - \langle X, Y\rangle F_k + \langle Y, F_k\rangle X,$$

where X, Y are arbitrary smooth vector fields on M. This connection is not symmetric, it is the unique metric connection with the torsion $T(X,Y) = \langle Y, F_k\rangle X - \langle X, F_k\rangle Y$, [P-W].

Note that the equations of a Gaussian thermostat define significantly different flows for different values of $k = v^2$, while the equations (11) scale so that for different values of v^2 we get the same flow up to a change of time by a constant factor. To get complete understanding of the dynamics of (10) we need to consider the dynamics of the geodesic flow of the metric connection (11) for the whole family of vector fields $F_k = \frac{1}{k}E, k > 0$.

If the manifold M is the locally homogeneous space of Section 1 and the vector field E is left invariant, then we obtain a family of left-invariant para-metric connections. To get a weakly $\mathcal{K}$-invariant connection we have to have $KE = E$, i.e., E must be orthogonal to $\mathfrak{g}_0$. The equations of geodesics for these connections give us a family of

quadratic $L-F$ systems. Assuming that $E^2=1$, we get that in this case the respective vector field in $\mathfrak{g}_0$ is equal to

$$F(\xi) = L^*\xi - \frac{1}{k}\xi.$$

We obtain the following corollary of Theorem 6.2 and the considerations leading to Theorem 9.2.

Corollary 10.1. *Let r_{min} and r_{max} be the smallest and the largest real parts of the eigenvalues of L.*

If $r_{min} < r_{max}$ then for $k = v^2$ such that $r_{min} < \frac{1}{k} < r_{max}$ the Gaussian thermostat (10) in M is integrable.

For $k = v^2$ such that $\frac{1}{k} > r_{max}$ the trajectories of the Gaussian thermostat (10) in the compact space N are asymptotic to a subsystem, which is the suspension of the automorphism e^L.

In particular if $r_{min} < 0 < r_{max}$ then for $0 < k < \frac{1}{r_{max}}$ the Gaussian thermostat (10), for the field E orthogonal to $\mathfrak{g}_0$, has a global attractor carrying the suspension of e^L (which can be an Anosov flow). However for $k > \frac{1}{r_{max}}$ it is integrable. This is a somewhat paradoxical behavior: strong mixing properties at low kinetic energy, and quasi-periodic motions for high kinetic energy. Admittedly the strong mixing properties occur at the asymptotic submanifold of half the dimension of the phase space. And this subsystem is present for all values of the kinetic energy, but it is not an attractor or a repellor in the integrable case.

11. A family of Riemannian metrics with integrable geodesic flows

The Gaussian thermostats can be associated with Weyl connections, [W1],[W2]. The left invariant vector field E of Section 10 is a gradient of a function on the group G, which factors to $M = \Gamma_0\backslash G$. Indeed we have $E = \frac{\partial}{\partial u} = \nabla u$. The Weyl connection defined by gradient vector fields are Levi-Civita connections of modified metrics, namely for the vector field $F_k = \frac{1}{k}E$ we need to multiply the initial left invariant metric by the function $e^{\frac{2u}{k}}$.

Let for simplicity L be diagonal with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. The left invariant metric on G is given by

$$ds^2 = e^{-2u\lambda_1}dw_1^2 + e^{-2u\lambda_2}dw_2^2 + \cdots + e^{-2u\lambda_n}dw_n^2 + du^2.$$

Introducing the new variables

$$x_0 = e^{\frac{u}{k}}, x_i = \frac{1}{k}w_i, i = 1, \dots, n,$$

we obtain

$$e^{\frac{2u}{k}}ds^2 = k^2\left(x_0^{2\tau_1}dx_1^2 + x_0^{2\tau_2}dx_2^2 + \cdots + x_0^{2\tau_n}dx_n^2 + dx_0^2\right),$$

where $\tau_i = 1 - k\lambda_i, i = 1, \dots, n$. It was calculated in [T-W] that the Weyl sectional curvatures are non-positive in this case if and only if

$\tau_i \geq 1$ or $\tau_i = 0$ for $i = 1, \ldots, n$. The sign of sectional curvatures of the last metric must be the same as for the Weyl connection ([P-W]).

Considering the geodesic flow of the Riemannian metric on $M = \mathbb{T}^n \times \mathbb{R}_+ = \{(x; x_0) | x \ mod \ \Gamma_0, x_0 > 0\}$, where Γ_0 is a lattice in $\mathbb{R}^n$, we conclude that it is integrable if and only if there are exponents τ_i of opposite signs. It follows from the considerations of Section 10, but this phenomenon can be greatly generalized. Note that the metric is not complete since the variable x_0 is assumed positive. Nevertheless in the integrable case geodesics from an open and dense subset can be extended indefinitely since they do not leave a compact subset of M. If the conditions of integrability are not satisfied then almost all geodesics leave every compact subset of M.

For an open interval $(a, b) \subset \mathbb{R}$, finite or infinite, let us consider a Riemannian metric on

$M = \mathbb{T}^n \times (a, b) = \{(x; x_0) | x \ mod \ \Gamma_0, a < x_0 < b\}$,

$$ds^2 = \alpha_1^2 dx_1^2 + \alpha_2^2 dx_2^2 + \cdots + \alpha_n^2 dx_n^2 + dx_0^2,$$

where all the functions α_i are positive functions of the variable x_0 alone, defined on the interval (a, b).

Proposition 11.1. *If* $\lim_{x_0 \to a} \alpha_i(x_0) = 0$ *and* $\lim_{x_0 \to b} \alpha_l(x_0) = 0$ *for some* i *and* l *then the geodesic flow is integrable.*

Proof. Let us pass to the hamiltonian formulation. We get the momenta

$$p_0 = \dot{x}_0, \ p_j = \alpha_j^2 \dot{x}_j, j = 1, \ldots, n,$$

and the hamiltonian

$$H = \frac{1}{2}\left(p_0^2 + \sum_{j=1}^{n} \alpha_j^{-2} p_j^2\right).$$

We have the obvious n first integrals $p_1, \ldots, p_n$. Together with H we get $n+1$ first integrals in involution. Under the assumption that $p_i \neq 0$ and $p_l \neq 0$ the common level set of the first integrals must be compact in the variable x_0, hence altogether compact. Indeed, we have

$$H \geq \alpha_l^{-2} p_i^2 + \alpha_l^{-2} p_l^2 \to +\infty \quad \text{as} \quad x_0 \to a, b.$$

By the Arnold-Liouville Theorem all regular compact level sets are tori carrying the quasi-periodic motions. □

This last Proposition takes us away from homogeneous systems, now we need only the $\mathbb{T}^n$ symmetry.

References

[A] Arnold, V.I.: Mathematical Methods of Classical Mechanics, Springer, New York, (1989)

[C-H-P] Chen, J.; Hu, H.; Pesin, Y.: The essential coexistence phenomenon in dynamics. Dyn. Syst. 28, 453-472 (2013)

[Bo1] O. I. Bogoyavlenskij, O.I.: A concept of integrability of dynamical systems. C. R. Math. Rep. Acad. Sci. Canada 18(4), 163-168 (1996)

[Bo2] O. I. Bogoyavlenskij, O.I.: Extended integrability and bi-Hamiltonian systems. Commun. Math. Phys. 196(1), 19-51 (1998)

[Bu1] Butler, L.: A new class of homogeneous manifolds with Liouville-integrable geodesic flows. C. R. Math. Acad. Sci. Soc. R. Can. 21, 127-131 (1999)

[Bu2] Butler, L.: Integrable hamiltonian flows with positive Lebesgue-measure entropy. Ergodic Theory Dyn. Syst. 23, 1671-1690 (2003)

[Bu3] Butler, L.: An integrable, volume-preserving flow on S2S3 with positive Lebesgue-measure entropy. Topological methods in the theory of integrable systems, 81-87, Camb. Sci. Publ., Cambridge (2006)

[Bu-P] Butler, L.; Paternain, G. P.: Magnetic flows on Sol-manifolds: dynamical and symplectic aspects. Comm. Math. Phys. 284, 187-202 (2008).

[B-T] Bolsinov, A.V., Taimanov, I.A.: Integrable geodesic flows with positive topological entropy. Invent. Math. 140, 639-650 (2000)

[D] Devaney, R.L.: Reversible diffeomorphisms and flows. Trans. Am. Math. Soc. 218, 89–113 (1976)

[DeV] DeVogelaere, R.: On the structure of periodic solutions of conservative systems, with applications. In: Lefschetz, S. (Ed.), Contribution to the Theory of Nonlinear Oscillations, vol. 4. Princeton Univ. Press, 53–84 (1958)

[F] Fasso, F.: Quasi-periodicity of motions and complete integrability of Hamiltonian systems. Ergod. Th. Dynam. Sys. 18(6), 1349-1362 (1998)

[G-R] Gallavotti, G., Ruelle, D.: SRB states and nonequilibrium statistical mechanics close to equilibrium. Commun. Math. Phys. 190, 279-285 (1997)

[H] Hoover, W.G.: Molecular Dynamics, Lecture Notes in Phys. 258, Springer (1986)

[L-R] Lamb, J. S. W.; Roberts, J. A. G.: Time-reversal symmetry in dynamical systems: a survey. Phys. D 112, 1-39 (1998).

[K-H] Katok, A.; Hasselblatt, B.: Introduction to the modern theory of dynamical systems. Cambridge Univ. Press, (1995)

[Mi] Milnor, J.: Curvatures of left invariant metrics on Lie groups. Advances in Math. 21 , no. 3, 293-329 (1976)

[M] Moser, J.: Convergent series expansions for quasi-periodic motions, Math. Ann. 169, 136–176 (1967)

[P-W] Przytycki, P.; Wojtkowski, M. P.: Gaussian thermostats as geodesic flows of nonsymmetric linear connections. Comm. Math. Phys. 277, 759-769 (2008)

[S] Sevryuk, M.B.: The finite dimensional reversible KAM theory. Physica D 112, 132–147 (1998)

[T-W] Tereszkiewicz, G; Wojtkowski, M.P.: Homogenous Weyl connections of non-positive curvature. arXiv: 1506.08176 (2015)

[W1] Wojtkowski, M.P.: W-flows on Weyl manifolds and Gaussian thermostats. J. Math. Pures Appl. 79(10), 953-974 (2000)

[W2] Wojtkowski, M.P.: Weyl manifolds and Gausssian thermostats. In: Proc. ICM Beijing 2002, Beijing: Higher Education Press, 2003, pp. 3511-3523

Department of Mathematics Physics and Computer Science
University of Opole
45-052 Opole, POLAND

E-mail address: `mpwojtkowski@math.uni.opole.pl`